Linear Algebra I

Richard Earl

Michaelmas Term 2021

0. PRELIMINARY MATERIAL

0.1 Syllabus

Systems of linear equations. Matrices and the beginnings of matrix algebra. Use of matrices to describe systems of linear equations. Elementary Row Operations (EROs) on matrices. Reduction of matrices to echelon form. Application to the solution of systems of linear equations. [2.5]

Inverse of a square matrix. The use of EROs to compute inverses; computational efficiency of the method. Transpose of a matrix; orthogonal matrices. [1]

Vector spaces: definition of a vector space over a field (such as $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$). Subspaces. Many explicit examples of vector spaces and subspaces. [1.5]

Span of a set of vectors. Examples such as row space and column space of a matrix. Linear dependence and independence. Bases of vector spaces; examples. The Steinitz Exchange Lemma; dimension. Application to matrices: row space and column space, row rank and column rank. Coordinates associated with a basis of a vector space. [2]

Use of EROs to find bases of subspaces. Sums and intersections of subspaces; the dimension formula. Direct sums of subspaces. [1.5]

Linear transformations: definition and examples (including projections associated with direct-sum decompositions). Some algebra of linear transformations; inverses. Kernel and image, Rank-Nullity Theorem. Applications including algebraic characterisation of projections (as idempotent linear transformations). [2]

Matrix of a linear transformation with respect to bases. Change of Bases Theorem. Applications including proof that row rank and column rank of a matrix are equal. [2]

Bilinear forms; real inner product spaces; examples. Mention of complex inner product spaces. Cauchy–Schwarz inequality. Distance and angle. The importance of orthogonal matrices. [1.5]

0.2 Reading list

(1) Gilbert Strang, Introduction to linear algebra (Fifth edition, Wellesley-Cambridge 2016). http://math.mit.edu/~gs/linearalgebra/
(2) T.S. Blyth and E.F. Robertson, Basic linear algebra (Springer, London, 1998).

Further Reading:
(3) Richard Kaye and Robert Wilson, Linear algebra (OUP, Oxford 1998), Chapters 1-5 and 8.
(4) Charles W. Curtis, Linear algebra - an introductory approach (Springer, London, Fourth edition, reprinted 1994).
(5) R. B. J. T. Allenby, Linear algebra (Arnold, London, 1995).
(6) D. A. Towers, A guide to linear algebra (Macmillan, Basingstoke, 1988).
(7) Seymour Lipschutz and Marc Lipson, Schaum's outline of linear algebra (McGraw Hill, New York & London, Fifth edition, 2013).

1. LINEAR SYSTEMS AND MATRICES

1.1 Systems of linear equations

Definition 1 *(a) By a **linear system**, or **linear system of equations**, we will mean a set of m simultaneous equations in n real variables $x_1, x_2, \ldots, x_n$ which are of the form*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1; \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2; \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array} \tag{1.1}$$

where a_{ij} and b_i are real constants.

*(b) Any vector $(x_1, x_2, \ldots, x_n)$ which satisfies (1.1) is said to be a **solution**; if the linear system has one or more solutions then it is said to be **consistent**. The **general solution** to the system is any description of all the solutions of the system. We will see, in due course, that such linear systems can have zero, one or infinitely many solutions.*

*(c) We will often write the linear system (1.1) as the **augmented matrix** $(A \,|\, \mathbf{b})$ where*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

For now, we won't consider a matrix (such as A) or vector (such as $\mathbf{b}$) to be anything more than an array of numbers.

Consider as a first example the following linear system of 3 equations in 3 variables.

Example 2 *Determine the solutions (if any) to the following equations.*

$$3x + y - 2z = -2; \qquad x + y + z = 2; \qquad 2x + 4y + z = 0.$$

Solution. We can substitute $z = 2 - x - y$ from the second equation into the first and third to find

$$\begin{aligned} 3x + y - 2(2 - x - y) = 5x + 3y - 4 = -2 \quad &\Longrightarrow \quad 5x + 3y = 2; \\ 2x + 4y + (2 - x - y) = x + 3y + 2 = 0 \quad &\Longrightarrow \quad x + 3y = -2. \end{aligned}$$

Subtracting the second of these equations from the first gives $4x = 4$ and so we see

$$x = 1, \qquad y = (-2 - x)/3 = -1, \qquad z = 2 - x - y = 2. \tag{1.2}$$

Thus there is a unique solution $(x, y, z) = (1, -1, 2)$. We can verify easily that this is indeed a solution (just to check that the system contains no contradictory information elsewhere that we haven't used). ■

Whilst we solved the above rigorously – we showed of necessity $(1, -1, 2)$ was the only possible solution and then verified it is a solution – our approach was a little *ad hoc*; at least, it's not hard to appreciate that if we were presented with 1969 equations in 2021 variables then we would need a much more systematic approach to treat them – or more likely we would need to be more methodical while programming our computers to determine any solutions for us. We introduce such a process called *row-reduction* here.

We first improve the notation, writing the system as an augmented matrix.

$$\left(\begin{array}{ccc|c} 3 & 1 & -2 & -2 \\ 1 & 1 & 1 & 2 \\ 2 & 4 & 1 & 0 \end{array}\right). \tag{1.3}$$

All that has been lost in this representation are the names of the variables, but these names are unchanging and unimportant in the actual handling of the equations. The advantages, we shall see, are that we will be able to progress systematically towards any solution and at each stage we shall retain all the information that the system contains – any redundancies (superfluous, unnecessary equations) or contradictions will naturally appear as part of the calculation.

This process is called *row-reduction.* It relies on three types of operation, called *elementary row operations* or *EROs*, which importantly do not affect the set of solutions of a linear system as we apply them.

Definition 3 *Given a linear system of equations, an* ***elementary row operation*** *or* ***ERO*** *is an operation of one of the following three kinds.*

(a) The ordering of two equations (or rows) may be swapped – for example, one might reorder the writing of the equations so that the first equation now appears third and vice versa.

(b) An equation may be multiplied by a non-zero scalar – for example, one might replace $2x - y + z = 3$ *by* $x - \frac{1}{2}y + \frac{1}{2}z = \frac{3}{2}$ *from multiplying both sides of the equation by* $\frac{1}{2}$.

(c) A multiple of one equation might be added to a different equation – for example, one might replace the second equation by the second equation plus twice the third equation.

Notation 4 *(a) Let* S_{ij} *denote the ERO which swaps rows* i *and* j *(or equivalently the* i*th and* j*th equations).*

(b) Let $M_i(\lambda)$ *denote the ERO which multiplies row* i *by* $\lambda \neq 0$ *(or equivalently both sides of the* i*th equation).*

(c) For $i \neq j$*, let* $A_{ij}(\lambda)$ *denote the ERO which adds* λ *times row* i *to row* j *(or does the same to the equations).*

Note this is not standard notation in any way, but I've introduced it here for convenience.

All these operations may well seem uncontroversial (their validity will be shown in Corollary 40) but it is probably not yet clear that these three simple operations are powerful enough to *reduce* any linear system to a point where any solutions can just be read off (Proposition 44, Theorem 47). Before treating the general case, we will see how the three equations in (1.3) can be solved using EROs to get an idea of the process.

Example 5 *Find all solutions of the linear system (1.3).*

Solution. If we use S_{12} to swap the first two rows the system becomes

$$\left(\begin{array}{ccc|c} 3 & 1 & -2 & -2 \\ 1 & 1 & 1 & 2 \\ 2 & 4 & 1 & 0 \end{array}\right) \xrightarrow{S_{12}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 1 & 0 \end{array}\right).$$

Now subtract three times the first row from the second, i.e. $A_{12}(-3)$ and follow this by subtracting twice the first row from the third, i.e. $A_{13}(-2)$, so that

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 1 & 0 \end{array}\right) \xrightarrow{A_{12}(-3)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -5 & -8 \\ 2 & 4 & 1 & 0 \end{array}\right) \xrightarrow{A_{13}(-2)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -5 & -8 \\ 0 & 2 & -1 & -4 \end{array}\right). \qquad (1.4)$$

We can now divide the second row by -2, i.e. $M_2(-1/2)$ to find

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -5 & -8 \\ 0 & 2 & -1 & -4 \end{array}\right) \xrightarrow{M_2(-1/2)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2\frac{1}{2} & 4 \\ 0 & 2 & -1 & -4 \end{array}\right).$$

We then subtract the second row from the first, i.e. $A_{21}(-1)$, and follow this by subtracting twice the second row from the third, i.e. $A_{23}(-2)$, to obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2\frac{1}{2} & 4 \\ 0 & 2 & -1 & -4 \end{array}\right) \xrightarrow{A_{21}(-1)} \left(\begin{array}{ccc|c} 1 & 0 & -1\frac{1}{2} & -2 \\ 0 & 1 & 2\frac{1}{2} & 4 \\ 0 & 2 & -1 & -4 \end{array}\right) \xrightarrow{A_{23}(-2)} \left(\begin{array}{ccc|c} 1 & 0 & -1\frac{1}{2} & -2 \\ 0 & 1 & 2\frac{1}{2} & 4 \\ 0 & 0 & -6 & -12 \end{array}\right). \qquad (1.5)$$

If we divide the third row by -6, i.e. $M_3(-1/6)$, the system becomes

$$\left(\begin{array}{ccc|c} 1 & 0 & -1\frac{1}{2} & -2 \\ 0 & 1 & 2\frac{1}{2} & 4 \\ 0 & 0 & -6 & -12 \end{array}\right) \xrightarrow{M_3(-1/6)} \left(\begin{array}{ccc|c} 1 & 0 & -1\frac{1}{2} & -2 \\ 0 & 1 & 2\frac{1}{2} & 4 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Finally, we subtract $2\frac{1}{2}$ times the third row from the second, i.e. $A_{32}(-2\frac{1}{2})$, and follow this by adding $1\frac{1}{2}$ times the third row to the first, i.e. $A_{31}(1\frac{1}{2})$.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1\frac{1}{2} & \text{-2} \\ 0 & 1 & 2\frac{1}{2} & 4 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{A_{32}(-5/2)} \left(\begin{array}{ccc|c} 1 & 0 & -1\frac{1}{2} & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{A_{31}(3/2)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

The rows of the final matrix represent the equations $x = 1$, $y = -1$, $z = 2$ as expected from (1.2). ■

Remark 6 *In case the systematic nature of the previous example isn't apparent, note that the first three operations* S_{12}, $A_{12}(-3)$, $A_{13}(-2)$ *were chosen so that the first column became* $(1, 0, 0)^T$ *in (1.4). There were many other ways to achieve this: for example, we could have begun with* $M_1(1/3)$ *to divide the first row by* 3, *then used* $A_{12}(-1)$ *and* $A_{13}(-2)$ *to clear the rest of the column. Once done, we then produced a similar leading entry of* 1 *in the second row with* $M_2(-1/2)$ *and used* $A_{21}(-1)$ *and* $A_{23}(-2)$ *to turn the second column into* $(0, 1, 0)^T$ *in (1.5). The final three EROs were chosen to transform the third column to* $(0, 0, 1)^T$ *at which point we could simply read off the solutions.*

Here are two slightly different examples, the first where we find that there are infinitely many solutions, whilst in the second example we see that there are no solutions.

Example 7 *Find the general solution of the following systems of equations in variables* x_1, x_2, x_3, x_4.

(a) $\quad x_1 - x_2 + x_3 + 3x_4 = 2; \qquad 2x_1 - x_2 + x_3 + 2x_4 = 4; \qquad 4x_1 - 3x_2 + 3x_3 + 8x_4 = 8.$

(b) $\quad x_1 + x_2 + x_3 + x_4 = 4; \qquad 2x_1 + 3x_2 - 2x_3 - 3x_4 = 1; \qquad x_1 + 5x_3 + 6x_4 = 1.$

Solution. (a) This time we will not spell out at quite so much length which EROs are being used. But we continue in a similar vein to the previous example and proceed by the method outlined in Remark 6.

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 2 & 4 \\ 4 & -3 & 3 & 8 & 8 \end{array}\right) \overset{\substack{A_{12}(-2)\\A_{13}(-4)}}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & -4 & 0 \end{array}\right) \overset{\substack{A_{21}(1)\\A_{23}(-1)}}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We have manipulated our system of three equations to two equations equivalent to the original system, namely

$$x_1 - x_4 = 2; \qquad x_2 - x_3 - 4x_4 = 0. \tag{1.6}$$

The presence of the zero row in the last matrix means that there was some redundancy in the system. Note, for example that the third equation can be deduced from the first two (it's the second equation added to twice the first) and so it provides no new information. As there are now only two equations in four variables, it's impossible for each column to contain a row's leading entry. In this example, the third and fourth columns lack such an entry. To describe all the solutions to a consistent system, we assign parameters to the columns/variables without leading entries. In this case that's x_3 and x_4 and we'll assign parameters by setting $x_3 = s$, $x_4 = t$, and then use the two equations in (1.6) to read off x_1 and x_2. So

$$x_1 = t + 2, \qquad x_2 = s + 4t, \qquad x_3 = s, \qquad x_4 = t, \tag{1.7}$$

or we could write

$$(x_1, x_2, x_3, x_4) = (t + 2, s + 4t, s, t) = (2, 0, 0, 0) + s(0, 1, 1, 0) + t(1, 4, 0, 1). \tag{1.8}$$

For each choice of s and t we have a solution as in (1.7) and this is one way of representing the general solution. (1.8) makes more apparent that these solutions form a plane in $\mathbb{R}^4$, a plane which passes through $(2, 0, 0, 0)$ is parallel to $(0, 1, 1, 0)$ and $(1, 4, 0, 1)$ with s, t parametrizing the plane.

(b) Applying EROs again in a like manner, we find

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 2 & 3 & -2 & -3 & 1 \\ 1 & 0 & 5 & 6 & 1 \end{array}\right) \overset{\substack{A_{12}(-2)\\A_{13}(-1)}}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & -4 & -5 & -7 \\ 0 & -1 & 4 & 5 & -3 \end{array}\right)$$

$$\overset{A_{23}(1)}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & -4 & -5 & -7 \\ 0 & 0 & 0 & 0 & -10 \end{array}\right) \overset{\substack{M_3(-1/10)\\A_{21}(-1)}}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & -5 & -6 & -11 \\ 0 & 1 & -4 & -5 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Note that any $\mathbf{x} = (x_1, x_2, x_3, x_4)$ which solves the final equation must satisfy

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1.$$

There clearly are no such x_i and so there are no solutions to this equation. Any solution to the system has, in particular, to solve the third equation and so this system has no solutions. In fact, this was all apparent once the third row had become $\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -10 \end{array}\right)$ as the equation it represents is clearly insolvable also. The final two EROs were simply done to put the matrix into what is called *reduced row echelon form* (see Definition 41). ■

Examples 5, 7(a) and 7(b) are specific examples of the following general cases.

- *A linear system can have no, one or infinitely many solutions.*

We shall prove this in due course (Proposition 44). We finish our examples though with a linear system that involves a parameter – so really we have a family of linear systems, one for each value of that parameter. What EROs may be permissible at a given stage may well depend on the value of the parameter and so we may see (as below) that such a family can exhibit all three of the possible scenarios just described.

Example 8 *Consider the system of equations in x, y, z,*

$$x + z = -5; \qquad 2x + \alpha y + 3z = -9; \qquad -x - \alpha y + \alpha z = \alpha^2,$$

where α is a constant. For which values of α has the system one solution, none or infinitely many?

Solution. Writing this system in matrix form and applying EROs we can argue as follows.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 2 & \alpha & 3 & -9 \\ -1 & -\alpha & \alpha & \alpha^2 \end{array}\right) \underset{A_{13}(1)}{\overset{A_{12}(-2)}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & \alpha & 1 & 1 \\ 0 & -\alpha & \alpha+1 & \alpha^2-5 \end{array}\right) \overset{A_{23}(1)}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & \alpha & 1 & 1 \\ 0 & 0 & \alpha+2 & \alpha^2-4 \end{array}\right). \tag{1.9}$$

At this point, which EROs are permissible depends on the value of α. We would like to divide the second equation by α and the third by $\alpha + 2$. Both these are permissible provided that $\alpha \neq 0$ and $\alpha \neq -2$. We will have to treat separately those particular cases but, assuming for now that $\alpha \neq 0, -2$, we obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & \alpha & 1 & 1 \\ 0 & 0 & \alpha+2 & \alpha^2-4 \end{array}\right) \underset{M_3(1/(\alpha+2))}{\overset{M_2(1/\alpha)}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 1 & 1/\alpha & 1/\alpha \\ 0 & 0 & 1 & \alpha-2 \end{array}\right) \underset{A_{32}(-1/\alpha)}{\overset{A_{31}(-1)}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\alpha-3 \\ 0 & 1 & 0 & 3/\alpha-1 \\ 0 & 0 & 1 & \alpha-2 \end{array}\right)$$

and we see that the system has a unique solution when $\alpha \neq 0, -2$. Returning though to the last matrix of (1.9) for our two special cases, we would proceed as follows.

$$\alpha = 0: \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -4 \end{array}\right) \underset{M_3(-1/6)}{\overset{A_{23}(-2)}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

$$\alpha = -2: \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{M_2(-1/2)}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -5 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We see then that the system is inconsistent when $\alpha = 0$ (because of the insolvability of the third equation) whilst there are infinitely many solutions $x = -5 - t$, $y = (t-1)/2$, $z = t$, when $\alpha = -2$. We assign a parameter, here t, to the variable z as the third column has no leading entry. ■

Before we treat linear systems more generally, we will first need to discuss matrices and their algebra.

1.2 Matrices and matrix algebra

At its simplest, a *matrix* is just a two-dimensional array of numbers; for example

$$\begin{pmatrix} 1 & 2 & -3 \\ \sqrt{2} & \pi & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1.2 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.10}$$

are all matrices. The examples above are respectively a 2×3 matrix, a 3×1 matrix and a 2×2 matrix (read '2 by 3' etc.); the first figure refers to the number of horizontal *rows* and the second to the number of vertical *columns* in the matrix. Row vectors in $\mathbb{R}^n$ are $1 \times n$ matrices and columns vectors in $\mathbb{R}^n_{\text{col}}$ are $n \times 1$ matrices.

Definition 9 *Let m, n be positive integers. An $m \times n$* ***matrix*** *is an array of real numbers arranged into m* ***rows*** *and n* ***columns****.*

Example 10 *Consider the first matrix above. Its second row is $\begin{pmatrix} \sqrt{2} & \pi & 0 \end{pmatrix}$ and its third column is $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.*

Definition 11 *The numbers in a matrix are its* ***entries****. Given an $m \times n$ matrix A, we will write a_{ij} for the entry in the ith row and jth column. Note that i can vary between 1 and m, and that j can vary between 1 and n. So*

$$i\text{th row} = (a_{i1}, \ldots, a_{in}) \qquad \text{and} \qquad j\text{th column} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Notation 12 *We shall denote the set of real $m \times n$ matrices as M_{mn}. Note that $M_{1n} = \mathbb{R}^n$ and that $M_{n1} = \mathbb{R}^n_{\text{col}}$.*

Example 13 *If we write A for the first matrix in (1.10) then we have $a_{23} = 0$ and $a_{12} = 2$.*

There are three important operations that can be performed with matrices: *matrix addition, scalar multiplication* and *matrix multiplication.* As with vectors, not all pairs of matrices can be meaningfully added or multiplied.

Definition 14 ***Addition*** *Let $A = (a_{ij})$ be an $m \times n$ matrix (recall: m rows and n columns) and $B = (b_{ij})$ be a $p \times q$ matrix. As with vectors, matrices are added by adding their corresponding entries. So, as with vectors, to add two matrices they have to be the same size – that is, to add A and B, we must have $m = p$ and $n = q$. If we write $C = A + B = (c_{ij})$ then*

$$c_{ij} = a_{ij} + b_{ij} \qquad \text{for } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n.$$

Example 15 *Let*

$$\underbrace{A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}}_{2\times 2}, \qquad \underbrace{B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}_{2\times 3}, \qquad \underbrace{C = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}}. \tag{1.11}$$

Of the possible sums involving these matrices, only $A + C$ and $C + A$ make sense as B is a different size. Note that

$$A + C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = C + A.$$

Remark 16 *In general, matrix addition is* ***commutative*** *as for matrices M and N of the same size we have*

$$M + N = N + M.$$

Addition of matrices is also ***associative*** *as*

$$L + (M + N) = (L + M) + N$$

for any matrices of the same size.

Definition 17 *The $m \times n$* ***zero matrix*** *is the matrix with m rows and n columns whose every entry is 0. This matrix is simply denoted as 0 unless we need to specify its size, in which case it is written 0_{mn}. For example,*

$$0_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple check shows that $A + 0_{mn} = A = 0_{mn} + A$ for any $m \times n$ matrix A.

Definition 18 ***Scalar Multiplication*** *Let $A = (a_{ij})$ be an $m \times n$ matrix and k be a real number (a scalar). Then the matrix kA is defined to be the $m \times n$ matrix with (i,j)th entry equal to ka_{ij}.*

Example 19 *Show that $2(A + B) = 2A + 2B$ for the following matrices:*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & -2 \\ 5 & 1 \end{pmatrix}.$$

Solution. Here we are checking the **distributive law** in a specific example. We note that

$$A + B = \begin{pmatrix} 1 & 0 \\ 8 & 5 \end{pmatrix}, \qquad \text{and so} \qquad 2(A + B) = \begin{pmatrix} 2 & 0 \\ 16 & 10 \end{pmatrix};$$

$$2A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}, \quad \text{and} \quad 2B = \begin{pmatrix} 0 & -4 \\ 10 & 2 \end{pmatrix}, \quad \text{so} \quad 2A + 2B = \begin{pmatrix} 2 & 0 \\ 16 & 10 \end{pmatrix}.$$

■

Remark 20 *More generally the following identities hold. Let A, B, C be $m \times n$ matrices and λ, μ be real numbers.*

$$\begin{array}{ccc} A + 0_{mn} = A; & A + B = B + A; & 0A = 0_{mn}; \\ A + (-A) = 0_{mn}; & (A+B)+C = A+(B+C); & 1A = A; \\ (\lambda+\mu)A = \lambda A + \mu A; & \lambda(A+B) = \lambda A + \lambda B; & \lambda(\mu A) = (\lambda\mu)A. \end{array}$$

These are readily verified and show that M_{mn} is a real vector space. ■

Based on how we added matrices then you might think that we multiply matrices in a similar fashion, namely multiplying corresponding entries, but we do not. At first glance the rule for multiplying matrices is going to seem rather odd but, in due course, we will see why matrix multiplication is done as follows and that this is natural in the context of matrices representing linear maps.

Definition 21 ***Matrix Multiplication*** *We can multiply an $m \times n$ matrix $A = (a_{ij})$ with an $p \times q$ matrix $B = (b_{ij})$ if $n = p$. That is, A must have as many columns as B has rows. If this is the case then the product $C = AB$ is the $m \times q$ matrix with entries*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \text{for } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant q. \tag{1.12}$$

It may help to write the rows of A as $\mathbf{r}_1, \ldots, \mathbf{r}_m$ and the columns of B as $\mathbf{c}_1, \ldots, \mathbf{c}_q$. Rule (1.12) then states that

$$\text{the } (i,j)\text{th entry of } AB = \mathbf{r}_i \cdot \mathbf{c}_j \qquad \text{for } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant q. \tag{1.13}$$

We dot (i.e. take the scalar product of) the rows of A with the columns of B; specifically to find the (i,j)th entry of AB we dot the ith row of A with the jth column of B.

Remark 22 *We shall give full details later as to why it makes sense (and, in fact, is quite natural) to multiply matrices as in (1.12). For now, it is worth noting the following. Let A be an $m \times n$ matrix and B be $n \times p$ so that AB is $m \times p$. There is a map L_A from $\mathbb{R}^n_{\text{col}}$ to $\mathbb{R}^m_{\text{col}}$ associated with A, as given an $n \times 1$ column vector $\mathbf{v}$ in $\mathbb{R}^n_{\text{col}}$ then $A\mathbf{v}$ is a $m \times 1$ column vector in $\mathbb{R}^m_{\text{col}}$. (Here the L denotes that we are multiplying on the left or premultiplying.) So we have associated maps*

$$L_A \text{ from } \mathbb{R}^n_{\text{col}} \text{ to } \mathbb{R}^m_{\text{col}}, \qquad L_B \text{ from } \mathbb{R}^p_{\text{col}} \text{ to } \mathbb{R}^n_{\text{col}}, \qquad L_{AB} \text{ from } \mathbb{R}^p_{\text{col}} \text{ to } \mathbb{R}^m_{\text{col}}.$$

Multiplying matrices as we have, it turns out that

$$L_{AB} = L_A \circ L_B.$$

This is equivalent to $(AB)\mathbf{v} = A(B\mathbf{v})$ which follows from the associativity of matrix multiplication. So matrix multiplication is best thought of as composition: performing L_{AB} is equal to the performing L_B then L_A. ■

Example 23 *Calculate the possible products of the pairs of matrices in (1.11).*

Solution. Recall that a matrix product MN makes sense if M has the same number of columns as N has rows. A, B, C are respectively 2×2, 2×3, 2×2 matrices. so the products we can form are AA, AB, AC, CA, CB, CC. Let's slowly go through the product AC.

$$\begin{pmatrix} \boxed{1} & \boxed{2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \boxed{1} & -1 \\ \boxed{1} & -1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 1 & ?? \\ ?? & ?? \end{pmatrix} = \begin{pmatrix} 3 & ?? \\ ?? & ?? \end{pmatrix}.$$

This is how we calculate the $(1,1)$th entry of AC. We take the first row of A and the first column of C and dot them together. We complete the remainder of the product as follows:

$$\begin{pmatrix} \boxed{1} & \boxed{2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \boxed{-1} \\ 1 & \boxed{-1} \end{pmatrix} = \begin{pmatrix} 2 & \boxed{1 \times (-1) + 2 \times (-1)} \\ ?? & ?? \end{pmatrix} = \begin{pmatrix} 3 & \boxed{-3} \\ ?? & ?? \end{pmatrix};$$

$$\begin{pmatrix} 1 & 2 \\ \boxed{-1} & \boxed{0} \end{pmatrix} \begin{pmatrix} \boxed{1} & -1 \\ \boxed{1} & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ \boxed{(-1) \times 1 + 0 \times 1} & ?? \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ \boxed{-1} & ?? \end{pmatrix};$$

$$\begin{pmatrix} 1 & 2 \\ \boxed{-1} & \boxed{0} \end{pmatrix} \begin{pmatrix} 1 & \boxed{-1} \\ 1 & \boxed{-1} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & \boxed{(-1) \times (-1) + 0 \times (-1)} \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -1 & \boxed{1} \end{pmatrix}.$$

So finally

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix}.$$

We complete the remaining examples more quickly but still leaving a middle stage in the calculation to help see the process.

$$\begin{aligned}
AA &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1-2 & 2+0 \\ -1+0 & -2+0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & -2 \end{pmatrix}; \\
AB &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+6 & 2+4 & 3+2 \\ -1+0 & -2+0 & -3+0 \end{pmatrix} = \begin{pmatrix} 7 & 6 & 5 \\ -1 & -2 & -3 \end{pmatrix}; \\
CA &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1+1 & 2-0 \\ 1+1 & 2-0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}; \\
CB &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1-3 & 2-2 & 3-1 \\ 1-3 & 2-2 & 3-1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{pmatrix}; \\
CC &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

■

Definition 24 *The* $n \times n$ ***identity matrix*** I_n *is the* $n \times n$ *matrix with entries*

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The identity matrix will be simply denoted as I *unless we need to specify its size. The* (i,j)*th entry of* I *is denoted as* δ_{ij} *which is referred to as the* ***Kronecker delta.***

Remark 25 ***(Sifting Property of the Kronecker Delta)*** *Let $x_1, \ldots, x_n$ be n real numbers, and $1 \leqslant k \leqslant n$. Then*

$$\sum_{i=1}^{n} x_i \delta_{ik} = x_k.$$

This is because $\delta_{ik} = 0$ when $i \neq k$ and $\delta_{kk} = 1$. Thus the above sum sifts out (i.e. selects) the kth element x_k. ■

There are certain important points to highlight from Example 23, some of which make matrix algebra crucially different from the algebra of real numbers.

Proposition 26 ***(Properties of Matrix Multiplication)*** *(a) For an $m \times n$ matrix A and positive integers l, p,*

$$A0_{np} = 0_{mp}; \qquad 0_{lm}A = 0_{ln}; \qquad AI_n = A; \qquad I_m A = A.$$

*(b) Matrix multiplication is **not commutative**; $AB \neq BA$ in general, even if both products meaningfully exist and have the same size.*
*(c) Matrix multiplication is **associative**; for matrices A, B, C, which are respectively $m \times n$, $n \times p$ and $p \times q$ we have*

$$A(BC) = (AB)C.$$

*(d) The **distributive** laws hold for matrix multiplication; whenever the following products and sums make sense,*

$$A(B + C) = AB + AC, \qquad \text{and} \qquad (A + B)C = AC + BC.$$

*(e) In Example 23 we saw $CC = 0$ even though $C \neq 0$ – so one **cannot** conclude from $MN = 0$ that either matrix M or N is zero.*

Proof. (a) To find an entry of the product $A0_{np}$ we dot a row of A with a zero column of 0_{np} and likewise in the product $0_{lm}A$ we are dotting with zero rows. Also, by the sifting property,

$$\begin{aligned} \text{the } (i,j)\text{th entry of } AI_n &= \textstyle\sum_{k=1}^{n} a_{ik}\delta_{kj} = a_{ij}; \\ \text{the } (i,j)\text{th entry of } I_n A &= \textstyle\sum_{k=1}^{n} \delta_{ik} a_{kj} = a_{ij}. \end{aligned}$$

(b) In Example 23, we saw that $AC \neq CA$. More generally, if A is $m \times n$ and B is $n \times p$ then the product AB exists but BA doesn't even make sense as a matrix product unless $m = p$.
(c) Given i, j in the ranges $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant q$, we see

$$\begin{aligned} \text{the } (i,j)\text{th entry of } (AB)C &= \sum_{r=1}^{p} \left(\sum_{s=1}^{n} a_{is} b_{sr} \right) c_{rj}; \\ \text{the } (i,j)\text{th entry of } A(BC) &= \sum_{s=1}^{n} a_{is} \left(\sum_{r=1}^{p} b_{sr} c_{rj} \right). \end{aligned}$$

These are equal as the order of finite sums may be swapped.
(d) This is left as an exercise. ■

Because matrix multiplication is not commutative, we need to be clearer than usual in what we might mean by a phrase like 'multiply by the matrix A'; typically we need to give some context as to whether we have multiplied on the left or on the right.

Definition 27 *Let A and M be matrices.*
*(a) To **premultiply** M by A is to form the product AM – i.e. premultiplication is multiplication on the left.*
*(b) To **postmultiply** M by A is to form the product MA – i.e. postmultiplication is multiplication on the right.*

Notation 28 *We write A^2 for the product AA and similarly, for n a positive integer, we write A^n for the product*

$$\underbrace{AA\cdots A}_{n \text{ times}}.$$

Note that A must be a square matrix for this to make sense. We also define $A^0 = I$. Note that $A^m A^n = A^{m+n}$ for natural numbers m, n. Given a polynomial $p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, then we define

$$p(A) = a_k A^k + a_{k-1}A^{k-1} + \cdots + a_1 A + a_0 I.$$

Example 29 *Let*

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.14}$$

Then $A^2 = I_2$ for any choice of α. Also there is no matrix C (with real or complex entries) such that $C^2 = B$. This shows that the idea of a square root is a much more complicated issue for matrices than for real or complex numbers. A square matrix may have none or many, even infinitely many, different square roots.

Solution. We note for any α that

$$A^2 = \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + (-\cos\alpha)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

To show B has no square roots, say a, b, c, d are real (or complex) numbers such that

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix}.$$

Looking at the $(2,1)$ entry, we see $c = 0$ or $a + d = 0$. But $a + d = 0$ contradicts $b(a+d) = 1$ from the $(1,2)$ entry and so $c = 0$. From the $(1,1)$ entry we see $a = 0$ and from the $(2,2)$ entry we see $d = 0$, but these lead to the same contradiction. ■

Let's look at a simple case of simultaneous equations: 2 linear equations in two variables, such as

$$ax + by = e; \qquad cx + dy = f. \tag{1.15}$$

Simple algebraic manipulations show that *typically* there is a unique solution (x, y) given by

$$x = \frac{de - bf}{ad - bc}; \qquad y = \frac{af - ce}{ad - bc}. \tag{1.16}$$

However if $ad - bc = 0$ then this solution is meaningless. It's probably easiest to appreciate geometrically why this is: the equations in (1.15) represent lines in the xy-plane with gradients $-a/b$ and $-c/d$ respectively, and hence the two lines are parallel if $ad - bc = 0$. (Notice that this is still the correct condition when $b = d = 0$ and the lines are parallel and vertical.) If the lines are parallel then there cannot be a unique solution.

We can represent the two scalar equations in (1.15) and (1.16) by a single vector equation in each case:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}; \tag{1.17}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}. \tag{1.18}$$

Equation (1.17) is just a rewriting of the linear system (1.15). Equation (1.18) is a similar rewriting of the unique solution found in (1.16) and something we *typically* can do. It also introduces us to the notion of the *inverse* of a matrix. Note that

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc) I_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1.19}$$

So if $ad - bc \neq 0$ and we set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

then $BA = I_2$ and $AB = I_2$.

Definition 30 *Let A be a square matrix. We say that B is an* ***inverse*** *of A if $BA = AB = I$. We refer to a matrix with an inverse as* ***invertible*** *and otherwise the matrix is said to be* ***singular****.*

Proposition 31 *(**Properties of Inverses**)*
*(a) (**Uniqueness**) If a square matrix A has an inverse, then it is unique. We write A^{-1} for this inverse.*
*(b) (**Product Rule**) If A, B are invertible $n \times n$ matrices then AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.*
*(c) (**Involution Rule**) If A is invertible then so is A^{-1} with $(A^{-1})^{-1} = A$.*

Proof. (a) Suppose B and C were two inverses for an $n \times n$ matrix A then

$$C = I_n C = (BA)C = B(AC) = BI_n = B$$

as matrix multiplication is associative. Part (b) is left as Sheet 1, Exercise S3. To verify (c) note that

$$\left(A^{-1}\right) A = A(A^{-1}) = I$$

and so $(A^{-1})^{-1} = A$ by uniqueness. ■

Definition 32 *If A is $m \times n$ and $BA = I_n$ then B is said to be a* ***left inverse****; if C satisfies $AC = I_m$ then C is said to be a* ***right inverse****.*

- If A is $m \times n$ where $m \neq n$ then A cannot have both left and right inverses. (This is non-trivial. We will prove this later.)

- If A, B are $n \times n$ matrices with $BA = I_n$ then, in fact, $AB = I_n$ (Proposition 167).

Inverses, in the 2×2 case, are a rather simple matter to deal with.

Proposition 33 *The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$ then*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Remark 34 *The scalar quantity $ad - bc$ is called the determinant of A, written* $\det A$*. It is a non-trivial fact to show that a square matrix is invertible if and only if its determinant is non-zero. This will be proved in Linear Algebra II next term.* ■

Proof. We have already seen in (1.19) that if $ad - bc \neq 0$ then $AA^{-1} = I_2 = A^{-1}A$. If however $ad - bc = 0$ then

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

satisfies $BA = 0$. If an inverse C for A existed then, by associativity, $0 = 0C = (BA)C = B(AC) = BI_2 = B$. So each of a, b, c and d would be zero and consequently $A = 0$ which contradicts $AC = I_2$. ■

We conclude this section with the following theorem. The proof demonstrates the power of the sigma-notation for matrix multiplication introduced in (1.12) and that of the Kronecker delta. In this proof we will make use of the *standard basis for matrices.*

Notation 35 *For I, J in the range $1 \leqslant I \leqslant m$, $1 \leqslant J \leqslant n$, we denote by E_{IJ} the $m \times n$ matrix with entry 1 in the Ith row and Jth column and 0s elsewhere. Then*

$$\text{the } (i,j)\text{th entry of } E_{IJ} = \delta_{Ii}\delta_{Jj}$$

as $\delta_{Ii}\delta_{Jj} = 0$ unless $i = I$ and $j = J$ in which case it is 1. These matrices form the ***standard basis*** *for M_{mn}.*

Theorem 36 *Let A be an $n \times n$ matrix such that $AM = MA$ for all $n \times n$ matrices M. i.e. A commutes with all $n \times n$ matrices. Then $A = \lambda I_n$ for some real number λ.*

Proof. As A commutes with every $n \times n$ matrix, then in particular it commutes with each of the n^2 basis matrices E_{IJ}. So the (i,j)th entry of AE_{IJ} equals that of $E_{IJ}A$ for every I, J, i, j. Using the sifting property

$$\begin{aligned} \text{the } (i,j)\text{th entry of } AE_{IJ} &= \sum_{k=1}^{n} a_{ik}\delta_{Ik}\delta_{Jj} = a_{iI}\delta_{Jj}; \\ \text{the } (i,j)\text{th entry of } E_{IJ}A &= \sum_{k=1}^{n} \delta_{Ii}\delta_{Jk}a_{kj} = \delta_{Ii}a_{Jj}. \end{aligned}$$

Hence for all I, J, i, j,

$$a_{iI}\delta_{Jj} = \delta_{Ii}a_{Jj}. \tag{1.20}$$

Let $i \neq j$. If we set $I = J = i$, then (1.20) becomes $0 = a_{ij}$ showing that the non-diagonal entries of A are zero. If we set $I = i$ and $J = j$, then (1.20) becomes $a_{ii} = a_{jj}$, which shows that all the diagonal entries of A are equal – call this shared value λ and we have shown $A = \lambda I_n$. This shows that any such M is necessarily of the form λI_n, and conversely such matrices do indeed commute with every other $n \times n$ matrix. ■

1.3 Reduced Row Echelon Form

Now looking to treat linear systems more generally, we will first show that the set of solutions of a linear system does not change under the application of EROs. We shall see that applying any ERO to a linear system $(A|\mathbf{b})$ is equivalent to premultiplying by an invertible *elementary* matrix E to obtain $(EA|E\mathbf{b})$, and it is the invertibility of elementary matrices that means the set of solutions remains unchanged when we apply EROs.

Proposition 37 *(**Elementary Matrices**) Let A be an $m \times n$ matrix. Applying any of the EROs S_{IJ}, $M_I(\lambda)$ and $A_{IJ}(\lambda)$ is equivalent to pre-multiplying A by certain matrices which we also denote as S_{IJ}, $M_I(\lambda)$ and $A_{IJ}(\lambda)$. Specifically these matrices have entries*

$$\begin{aligned}
\text{the } (i,j)\text{th entry of } S_{IJ} &= \begin{cases} 1 & i = j \neq I, J, \\ 1 & i = J,\, j = I, \\ 1 & i = I, j = J, \\ 0 & \text{otherwise.} \end{cases} \\
\text{the } (i,j)\text{th entry of } M_I(\lambda) &= \begin{cases} 1 & i = j \neq I, \\ \lambda & i = j = I, \\ 0 & \text{otherwise.} \end{cases} \\
\text{the } (i,j)\text{th entry of } A_{IJ}(\lambda) &= \begin{cases} 1 & i = j, \\ \lambda & i = J,\, j = I, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

*The above matrices are known as **elementary matrices**.*

Proof. The proof is left as an exercise. ■

Example 38 *When $m = 3$ we see*

$$S_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3(7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \qquad A_{31}(-2) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that these elementary matrices are the results of performing the corresponding EROs $S_{21}, M_3(7), A_{31}(-2)$ on the identity matrix I_3. This is generally true of elementary matrices.

Proposition 39 *Elementary matrices are invertible.*

Proof. This follows from noting that

$$(S_{ij})^{-1} = S_{ji} = S_{ij}; \qquad (A_{ij}(\lambda))^{-1} = A_{ij}(-\lambda); \qquad (M_i(\lambda))^{-1} = M_i(\lambda^{-1}),$$

whether considered as EROs or their corresponding matrices. ■

Corollary 40 ***(Invariance of Solution Space under EROs)*** *Let $(A|\mathbf{b})$ be a linear system of m equations and E an elementary $m \times m$ matrix. Then $\mathbf{x}$ is a solution of $(A|\mathbf{b})$ if and only if $\mathbf{x}$ is a solution of $(EA|E\mathbf{b})$.*

Proof. The important point here is that E is invertible. So if $A\mathbf{x} = \mathbf{b}$ then $EA\mathbf{x} = E\mathbf{b}$ follows by premultiplying by E. But likewise if $EA\mathbf{x} = E\mathbf{b}$ is true then it follows that $A\mathbf{x} = \mathbf{b}$ by premultiplying by E^{-1}. ■

So applying an ERO, or any succession of EROs, won't alter the set of solutions of a linear system. The next key result is that, systematically using EROs, it is possible to reduce any system $(A|\mathbf{b})$ to *reduced row echelon form.* Once in this form it is simple to read off the system's solutions.

Definition 41 *A matrix A is said to be in* ***reduced row echelon form*** *(or simply* ***RRE form****) if*

(a) the first non-zero entry of any non-zero row is 1;

(b) in a column that contains such a leading 1, all other entries are zero;

(c) the leading 1 of a non-zero row appears to the right of the leading 1s of the rows above it;

(d) any zero rows appear below the non-zero rows.

Definition 42 *The process of applying EROs to transform a matrix into RRE form is called* ***row-reduction****, or just simply* ***reduction.*** *It is also commonly referred to as* ***Gauss-Jordan elimination****.*

Example 43 *Of the following matrices*

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -4 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & \sqrt{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix},$$

the first three are in RRE form. The fourth is not as the second column contains a leading 1 but not all other entries of that column are 0. The fifth matrix is not in RRE form as the leading entry of the third row is not 1.

We have yet to show that any matrix can be uniquely put into RRE form using EROs (Theorem 122) but – as we have already seen examples covering the range of possibilities – it seems timely to prove the following result here.

Proposition 44 ***(Solving Systems in RRE Form)*** *Let $(A|\mathbf{b})$ be a matrix in RRE form which represents a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n variables. Then*

(a) the system has no solutions if and only if the last non-zero row of $(A|\mathbf{b})$ is

$$\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array} \right).$$

(b) the system has a unique solution if and only if the non-zero rows of A form the identity matrix I_n. In particular, this case is only possible if $m \geqslant n$.

(c) the system has infinitely many solutions if $(A|\mathbf{b})$ has as many non-zero rows as A, and not every column of A contains a leading 1. The set of solutions can be described with k parameters where k is the number of columns not containing a leading 1.

Proof. If $(A|\mathbf{b})$ contains the row $\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array} \right)$ then the system is certainly inconsistent as no $\mathbf{x}$ satisfies the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = 1.$$

As $(A|\mathbf{b})$ is in RRE form, then this is the only way in which $(A|\mathbf{b})$ can have more non-zero rows than A. We will show that whenever $(A|\mathbf{b})$ has as many non-zero rows as A then the system $(A|\mathbf{b})$ is consistent.

Say, then, that both $(A|\mathbf{b})$ and A have r non-zero rows, so there are r leading 1s within these rows and we have $k = n - r$ columns without leading 1s. By reordering the numbering of the variables $x_1, \ldots, x_n$ if necessary, we can assume that the leading 1s appear in the first r columns. So, ignoring any zero rows, and remembering the system is in RRE form, the system now reads as the r equations:

$$x_1 + a_{1(r+1)}x_{r+1} + \cdots + a_{1n}x_n = b_1; \qquad \cdots \qquad x_r + a_{r(r+1)}x_{r+1} + \cdots + a_{rn}x_n = b_r.$$

We can see that if we assign $x_{r+1}, \ldots, x_n$ the k parameters $s_{r+1}, \ldots, s_n$, then we can read off from the r equations the values for $x_1, \ldots, x_r$. So for any values of the parameters we have a solution $\mathbf{x}$. Conversely though if $\mathbf{x} = (x_1, \ldots, x_n)$ is a solution, then it appears amongst the solutions we've just found when we assign values $s_{r+1} = x_{r+1}, \ldots, s_n = x_n$ to the parameters. We see that we have an infinite set of solutions associated with $k = n - r$ independent parameters when $n > r$ and a unique solution when $r = n$, in which case the non-zero rows of A are the matrix I_n. ■

Remark 45 *Note we showed in this proof that*

- *a system $(A|\mathbf{b})$ in RRE form is consistent if and only if $(A|\mathbf{b})$ has as many non-zero rows as A;*

- *all the solutions of a consistent system can be found by assigning parameters to the variables corresponding to the columns without leading 1s.* ■

Example 46

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right), \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

no solutions *unique solution*

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right), \quad \left(\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

one parameter family of solutions $(3-2s, s, 2, 1)$ *two parameter family of solutions* $(3+2s-2t, s, -2-t, t)$

Theorem 47 ***(Existence of RRE Form)***
Every $m \times n$ matrix A can be reduced by EROs to a matrix in RRE form.

Proof. Note that a $1 \times n$ matrix is either zero or can be put into RRE form by dividing by its leading entry. Suppose, as our inductive hypothesis, that any matrix with fewer than m rows can be transformed with EROs into RRE form. Let A be an $m \times n$ matrix. If A is the zero matrix, then it is already in RRE form. Otherwise there is a first column $\mathbf{c}_j$ which contains a non-zero element α. With an ERO we can swap the row containing α with the first row and then divide the first row by $\alpha \neq 0$ so that the $(1, j)$th entry now equals 1. Our matrix now takes the form

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & \tilde{a}_{1(j+1)} & \cdots & \tilde{a}_{1n} \\ 0 & \cdots & 0 & \tilde{a}_{2j} & \vdots & \vdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \tilde{a}_{mj} & \tilde{a}_{m(j+1)} & \cdots & \tilde{a}_{mn} \end{pmatrix},$$

for some new entries $\tilde{a}_{1(j+1)}, \ldots, \tilde{a}_{mn}$. Applying consecutively $A_{12}(-\tilde{a}_{2j}),\ A_{13}(-\tilde{a}_{3j}), \ldots, A_{1m}(-\tilde{a}_{mj})$ leaves column $\mathbf{c}_j = \mathbf{e}_1^T$ so that our matrix has become

$$\left(\begin{array}{cccc|ccc} 0 & \cdots & 0 & 1 & \tilde{a}_{1(j+1)} & \cdots & \tilde{a}_{1n} \\ \hline 0 & \cdots & 0 & 0 & & & \\ \vdots & \cdots & \vdots & \vdots & & B & \\ 0 & \cdots & 0 & 0 & & & \end{array}\right).$$

By induction, the $(m-1) \times (n-j)$ matrix B can be put into some RRE form by means of EROs. Applying these same EROs to the bottom $m-1$ rows of the above matrix we would have reduced A to

$$\left(\begin{array}{cccc|ccc} 0 & \cdots & 0 & 1 & \tilde{a}_{1(j+1)} & \cdots & \tilde{a}_{1n} \\ \hline 0 & \cdots & 0 & 0 & & & \\ \vdots & \cdots & \vdots & \vdots & & \mathrm{RRE}(B) & \\ 0 & \cdots & 0 & 0 & & & \end{array}\right).$$

To get the above matrix into RRE form we need to make zero any of $\tilde{a}_{1(j+1)}, \ldots, \tilde{a}_{1n}$ which are above a leading 1 in RRE(B); if $\tilde{a}_{1k}$ is the first such entry to lie above a leading 1 in row l then $A_{l1}(-\tilde{a}_{1k})$ will make the required edit and in due course we will have transformed A into RRE form. The result follows by induction. ■